MIE DOCTORAL QUALIFYING EXAMINATIONS GUIDELINES

Important information, deadlines, exams selection procedures

Written Comprehensive Examinations

2018
Northeastern University
Department of Mechanical and Industrial Engineering
DOCTORAL QUALIFYING EXAMINATIONS

Background and Motivation: To demonstrate breadth and depth in each of the subject exams, cross-over and merging exams are necessary in an effort to provide students with an opportunity to master the core disciplines in mechanical or industrial engineering (at both undergraduate and graduate levels) along with a focus area of importance to their specialization. These exams also provide an assessment as to whether students have adequate knowledge to pursue advanced study, and possess attributes of a doctoral candidate by demonstrating understanding of and the ability to apply fundamental principles. Also, an oral exam tied to the written exams is necessary in an effort to evaluate student’s potential to perform independent research in the chosen field of specialization for the doctoral program.

Doctoral Qualifying Examinations Framework: The Doctoral Qualifying Examinations consist of the following two parts:

1. Two Written Comprehensive exams, which are respectively referred to as Exam A and Exam B.
2. An Oral Area exam equivalently referred to as Area Exam. This exam can be administered at any time after passing the written comprehensive exams, but no later than the end of the semester in which the written exams are taken and passed.

WRITTEN COMPREHENSIVE EXAMINATIONS

All doctoral students who hold a master’s degree must take the written comprehensive exams no later than the first time that it is offered after their first academic year of study. Those admitted directly with a bachelor’s degree must take the written comprehensive exams no later than the first time that it is offered after their first two years of study. The written comprehensive exams include 2 exams, Exam A and Exam B; and are given on Thursday and Friday of the first week of classes during regular semesters.

Written Comprehensive Exams Rules: Exam A, about 4-6 hours in length, should be selected from the list of major exams based on the student’s concentration (i.e., Industrial Engineering, Materials, Mechanics, Mechatronics, or Thermofluids), see Table 1. No deviation from this rule will be permitted. As listed in Table 1, Exam B, about 1-2 hours in length, is a degree program-dependent exam and should be selected from the list of Exams B for each PhD program in MIE Department (i.e., PhD degree program in Industrial Engineering—IE or PhD degree program in Mechanical Engineering—ME). Only one exam from this list should be selected. All students are required to have their Research Advisor’s approval on selection of Exam B prior to registering to take the written comprehensive exams. Note that Exam B cannot be similar or close to one of the topics covered in Exam A.
ORAL AREA EXAMINATION

The objective of the oral exam, referred to as Area Exam, is to assess the student’s potential to perform independent research in the chosen field of specialization. This exam can be administered at any time after passing the written comprehensive exams, but no later than the end of the semester in which the written exams are taken and passed.

**Oral Area Examination Procedure:** The student’s Research Advisor convenes and chairs an Oral Examination Committee comprised of a minimum of three faculty or affiliated faculty members of the MIE Department deemed appropriate to the student’s research field. This committee provides a set of technical papers pertinent to the student’s research area. The Oral Examination Committee will then conduct the area exam that comprises the following two parts (both in one 1-hour session):

1. an oral presentation of 30-min on a select number of papers out of the assigned technical papers, and
2. an oral exam of 30-min by committee members’ questions and evaluation of the student covering topics specifically related to the student’s research area.

**GRADING PROCEDURE**

**Grading Procedure and Results of the Written Comprehensive Examination:** The MIE Graduate Affairs Committee (GAC) will review all students’ performance in the written comprehensive exams. Depending on the results of both major and minor exams and in consultation with the student’s Research Advisor, the GAC will recommend one of following three possible options:

1. *No invitation to oral area exam:* The student will be dismissed from the program. He/she may be granted an MS degree if the requirements are already met; otherwise, the student may continue to fulfill the requirements for an MS degree in industrial engineering (IE), mechanical engineering (ME), or operations research (OR).
2. *No invitation to oral area exam yet:* The student will be asked to re-take the written exam(s) again in the next offering; and/or take additional courses.
3. *Student is invited to oral area exam.*

**Grading Procedure and Results of the Oral Area Examination:** If the performance of the student in oral area exam is not satisfactory, the student will be dismissed from the program. He/she may be granted an MS degree if the requirements are met; otherwise, the student may continue to fulfill the requirements for an MS degree in industrial engineering (IE), mechanical engineering (ME), or operations research (OR).

Upon successfully passing the oral exam, the student continues in the PhD program and in case of passing all the required coursework, he/she will become a PhD Candidate. The results of the written and oral exams and any recommended coursework become part of the student’s record.
# Detailed List of Exams A and B

**Table 1:** List of Exams A* and B based on student’s concentration.

<table>
<thead>
<tr>
<th>Concentration</th>
<th>Exam A</th>
<th>Exam B</th>
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<tr>
<td>Industrial</td>
<td>Industrial Engineering (IND)</td>
<td><strong>Exams B for IE PhD Students:</strong></td>
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<td>1. Data Mining (DMN)</td>
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<td>2. Human-Machine Systems (HMS)</td>
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<td>5. Reliability and Quality Assurance (RQA)</td>
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<td>6. Supply Chain Engineering (SCE)</td>
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<tr>
<td>Materials</td>
<td>Materials Science Engineering (MSE)</td>
<td><strong>Exams B for ME PhD Students:</strong></td>
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<td>1. Control Systems (DSC3)</td>
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<td>4. Engineering Mathematics (MTH)</td>
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<td>5. Finite Element Method (MEC3)</td>
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<td>6. Fluid Mechanics (TFS2)</td>
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<td>7. Heat Transfer (TFS3)</td>
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<td>8. Kinetics of Materials (MSE1)</td>
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<td>9. Mechanics of Deformable Media (MEC1)</td>
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<td>11. Thermodynamics (TFS1)</td>
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<td>12. Thermodynamics of Materials (MSE2)</td>
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<td>Mechanics</td>
<td>Mechanics (MEC)</td>
<td><strong>List of Exams A</strong></td>
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<td>• Industrial Engineering (IND): Probability (IND1), Statistics (IND2), and Deterministic OR (IND3).</td>
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<td>• Materials Science Engineering (MSE): Kinetics of Materials (MSE1), Thermodynamics of Materials (MSE2); and Process, Structure, Property, and Performance of Materials (MSE3)</td>
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<td>• Mechanics (MEC): Mechanics of Deformable Media (MEC1), Dynamics and Vibration (MEC2), and Finite Element Method (MEC3)</td>
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<td>• Dynamic Systems and Control (DSC): Dynamic Systems (DSC1); Mechanical Vibrations (DSC2); and Control Systems (DSC3)</td>
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<td>• Thermofluids Science (TFS): Thermodynamics (TFS1); Fluid Mechanics (TFS2); and Heat Transfer (TFS3)</td>
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<td>Mechatronics</td>
<td>Dynamic Systems and Control (DSC)</td>
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<td>Thermofluids</td>
<td>Thermofluids Science (TFS)</td>
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Engineering Mathematics (MTH)

Topics Covered:

- Ordinary differential equations using exact methods, series and transforms.
- Partial differential equations using separation of variables (Fourier series, eigenfunction expansions) and transforms.
- Linear algebra matrices and linear equations, determinants, eigenvalue problems.
- Vector field theory including Cartesian, cylindrical and spherical coordinates, gradient, divergence and curl, and integral theorems (Divergence Theorem, Stokes’ Theorem).

Suggested References for Preparation:


**Previous Exam problems for MTH provided at the end of this document.**
Industrial Engineering (IND) – Exam A

**IND1: Probability:**
- Discrete and continuous random variables.
- Cumulative probability distributions and moment generating functions.
- Expectation of random variables.
- Discrete and continuous probability distributions including: binomial, Poisson, geometric, uniform, exponential and normal.
- Multivariable probability distributions, covariance and independence of random variables.
- Sampling distributions and limiting theorems.
- Parameter estimation.
- Confidence intervals and hypothesis testing.
- Regression and ANOVA.
- Chi-squared and non-parametric tests.

**IND2: Statistics:**
- Regression and ANOVA.
- Chi-squared and non-parametric tests.
- Stochastic Processes
- Poisson Process and Exponential Distribution
- Markov Chains (Discrete/Continuous time Markov Chain)
- Birth and Death process
- Queuing Theory

**IND3: Deterministic OR:**
- Linear Programming (LP)
  - Formulation of LP models
  - Solution of LP models with the graphical method and the simplex algorithm
  - Theory of the simplex method
  - Duality theory and dual simplex algorithm
  - Sensitivity Analysis in LP
  - The Transportation Problem and the Hungarian algorithm
- Network Optimization Models
  - The Shortest Path Problem
  - The Minimum Spanning Tree Problem
  - The Maximum Flow Problem
  - The Minimum Cost Flow Problem
- Dynamic Programming (DP)
  - Discrete State DP problems
  - Continuous State DP problems
Suggested References for Preparation:


Exams B for Industrial Engineering

Human-Machine Systems (HMS):

- Sociotechnical Systems and Human Systems Engineering, Human Capabilities and Characteristics.
- Engineering Anthropometry and Biomechanics.
- Physiology related to Human Factors and Workstation Design.
- Taxonomy of Biosensors for various cues (psychological, physiological, physical), states and behaviors of humans.
- Basic principles of biosensors, current technologies for building biosensors.
- Cognition and Information Processing, Decision-Making, Attention and Workload.
- Human-Machine Interface Design, Controls and Displays.
- Safety Engineering, Human Error and Accident analysis.
- Human Factors in Transportation, Automation.
- Human-Robot Interaction and human friendly mechatronics.

Suggested References for Preparation:


Manufacturing Systems (MFS):

- Manufacturing operations
  - Processing and assembly operations
  - Production facilities
  - Product-production relationships
- Manufacturing metrics and economics
  - Production performance metrics
  - Basic equations to estimate performance metrics
- Manufacturing cost
  - Computer numerical control
    - Fundamentals of NC technology
    - Computers and numerical control
    - Applications of NC technology
    - Analysis of positioning systems
  - Elements of manufacturing systems
    - Components of manufacturing systems
    - Types manufacturing systems
  - Single-station manufacturing cells
    - Single-station manned cells
    - Single-station automated cells
    - Applications of single-station cells
    - Analysis of single-station cells
  - Manual assembly lines
    - Fundamentals of manual assembly lines
    - Analysis of single-model assembly lines
    - Line balancing algorithms
    - Workstation details
    - Consideration in assembly line design
  - Automated production lines
    - Fundamental of automated production lines
    - Application of automated production lines
    - Analysis of transfer lines
    - Transfer line with internal storage
  - Automated assembly systems
    - Fundamentals of automated assembly systems
    - Analysis of automated assembly systems
  - Group Technology and cellular manufacturing
    - Part family and machining groups
    - Cellular manufacturing
    - Applications of group technology
    - Analysis of cellular manufacturing
  - Flexible Manufacturing cells and system
    - FMC/FMS components
    - FMS application considerations
    - Analysis of flexible manufacturing systems
  - Mechanical properties of materials
    - Strain-strain relationships
    - Hardness
    - Effect of temperature on properties
  - Fundamentals of metal forming
    - Basics of metal forming
    - Material behavior in metal forming
    - Temperature in metal forming
- Strain rate sensitivity
- Friction and lubrication in metal forming

- Bulk deformation processes in metalworking
  - Rolling and related deformation processes
  - Forging and related deformation processes
  - Extrusion
  - Wire and bar drawing

- Theory of metal machining
  - Overview of machining technology
  - Theory of chip formation in metal machining
  - Force relationships and the Merchant’s equation
  - Power and energy relationships in machining
  - Cutting temperature

- Machining operations and machine tools
  - Machining and part geometry
  - Turning and related operations
  - Drilling and related operations
  - Milling

- Cutting tool technology
  - Tool life
  - Tool geometry

**Suggested References for Preparation:**

Reliability and Quality Assurance (RQA):

- Quality planning, Control and Improvement.
- Process control, Discrete and Continuous Control Charts.
- Moving Average and Custom Control Charts.
- Discrete and Variable Sampling Methods, Mil Standards.
- Process Capability Analysis.
- Quality Engineering Method of Robust Design.
- Mathematical Definitions of Reliability, Hazard Rate, Intensity Function, Failure Rate and Availability.
- Stress and Strength Analysis, Reliability Block Design, Fault Tree Method.
- Network Reliability Methods, Markovian Methods, Reliability Testing.
  - Reliability Estimates from Field and Test Data.
  - Confidence Interval on Reliability.
- Maintenance and Replacement Policies.

Suggested References for Preparation:

Supply Chain Engineering (SCE):

- Forecasting
- Aggregate planning
- Sequencing and Scheduling
- Inventory analysis and control
- Materials requirement planning
- Pricing and revenue management
- Manufacturing resource planning
- Project management
- Contracts decisions
- Transportation decisions
- Location and distribution decisions
- Supplier selection methods
- Global supply chains

Suggested References for Preparation:


Materials Science Engineering (MSE)

MSE1: Kinetics of Materials:
- Diffusion
- Solidification
- Diffusional and diffusionless transformations in solids
- Molecular/colloidal forces.

MSE2: Thermodynamics of Materials:
- Three laws
- State functions
- Systems
- Phase equilibria and stability
- Behavior of Solutions
- Relations to phase diagrams
- Reactions among condensed phases and gases
- Statistical Thermo (Entropy, Heat capacity, etc.)

MSE3: Process, Structure, Property, and Performance of Materials:
- Crystal structures
- Dislocation theory
- Mechanical Behavior and strengthening mechanisms
- Fatigue and fracture
- Polymer processing and Properties
- Electronic and magnetic materials
- Environmental issues.

Suggested References for Preparation:
Mechanics (MEC)

MEC1: Mechanics of Deformable Bodies:
- Basic concepts of stress and strain and stress-strain relations.
- Yield strength and elastic-perfectly-plastic material behavior.
- Transformation of stress, principal stresses in three dimensions, Mohr’s circles.
- Boundary and continuity conditions for three-dimensional continua.
- Structural mechanics of bars, shafts, and beams under axial, torsional, and transverse loadings.
- Energy methods (including Castigliano’s Second Theorem) and calculation of deflections (including shear deformation) of beams, frames, and rings for statically determinate and statically indeterminate loadings.
- Thin-walled pressure vessels.
- Stability of structures; buckling of columns and structures.

MEC2: Dynamics and Vibration:
- Basic concepts of rigid body kinematics and kinetics.
- Newton’s Laws of Motion.
- Energy and momentum methods for particles and rigid bodies.
- Free and forced vibrations of single and multiple degree-of-freedom systems.
- Eigenvalue problems and modal expansions in vibration.
- Simple vibration of rods (longitudinal and torsional) and beams (bending).

MEC3: Finite Element Method:
- Weighted residual methods and identification of essential and natural boundary conditions.
- Implementation of variational methods, such as Rayleigh-Ritz, Galerkin to 1D boundary value problems.
- Derivation of interpolation functions and Cn continuity.
- Truss, beam, and 2D solid elements and element defects.
- Global stiffness matrix, assembly of element equations, numerical implementation of boundary conditions.
- Isoparametric elements and numerical integration.
- Solution of transient problems with implicit and explicit methods.
- Application of finite element method in heat transfer problems.

Suggested References for Preparation:

MEC1:

**MEC2:**


**MEC3:**


Dynamic Systems and Control (DSC)

DSC1: Dynamic Systems
- Dynamics of Mechanical Systems (Translational and Rotational)
- Electrical Circuits and Op-Amps
- Electromechanical Devices and DC Motors
- Thermal Systems Modeling
- Fluid and Level Systems Modeling
- Linearization Techniques
- Laplace Transforms and Application to Dynamic Systems Analysis
- 1st-order and 2nd-order Systems Response Characteristics: Time Constants; Natural Frequency, Damping Ratio, Damped Frequency; Impulse, Step and Ramp Responses; and Steady-state Error
- Approximating Higher-order Systems with equivalent 1st- and 2nd-order System with Dominant Modes

DSC2: Mechanical Vibrations
- Free and Forced Vibrations of Single Degree-Of-Freedom (SDOF) and Multiple Degree-Of-Freedom (MDOF) Systems
- Damped Vibrations
- General Eigenvalue Problem and Modal Analysis/Expansion in Vibrations
- Equations of Motion and Boundary Conditions for Transverse Vibrations of Strings; Longitudinal Vibrations of Bars; and Torsional Vibrations of Shafts
- Free and Forced Transverse (Bending) Vibrations of Euler-Bernoulli (Thin) Beams: Mode Shapes and Orthogonality Conditions

DSC3: Control Systems
- System Modeling Diagrams; Block Diagram Algebra and Reduction
- Effects of Poles/Zeros on System Response
- Routh’ Stability Criterion
- The 3-term PID (Proportional, Derivative and Integral) Control Design and Analysis
- Root Locus Plot
- System Type and Analysis of Steady-state Error
- Design via Root Locus and Feedback Compensation Techniques
- Bode Plots
- Nyquist Plot and Nyquist Stability Criterion
- Gain and Phase Margins
- Design via Frequency Response (Lead and Lag Compensations).
- Standard Forms (Input/Output) and State Equations
- State-Space, Observability, Controllability, Control Canonical Form, Pole Placement, Ackerman Formula
- Introduction to Real-time Control Implementation
Suggested References for Preparation:

**DSC1:**


**DSC2:**


**DSC3:**


Thermofluids Science (TFS)

**TFS1: Thermodynamics**
- Conservation of mass (steady and transient)
- Steady and Transient First and Second law of thermodynamics
- Energy, available energy and entropy
- Temperature and pressure
- Work and heat interaction
- Heat engine
- Characteristic function
- Simple system
- Equation of state
- Conversion devices
- Power generation
- Refrigeration and energy pump
- Chemical reaction and chemical equilibrium

**TFS2: Fluid Mechanics**
- Fluid Statics
- Finite control volume analyses
  - Reynolds transport theorem; continuity
  - Momentum Principle; conservation of energy
  - Bernoulli equation
- Differential control volume analyses
  - Navier-Stokes equations; laminar flow analyses
  - Boundary layer analyses; potential flow analyses
- Similitude
  - PI theorem; non-dimensional parameters
  - Model flow vs. prototype flow
  - Non-dimensionalization of governing equations and boundary conditions

**TFS3: Heat Transfer**
- Conduction
  - Steady and transient heat transfer in multi-dimensional systems (development and solution)
  - Problems involving internal heat generation source
  - Extended surface problems, including fin efficiency
  - Extended surface problems with varying cross section area, (Bessel Function Solutions)
  - Transient problems, lumped capacitance and multi-dimensional systems
- Convection
  - Definition of the heat transfer coefficient
  - Use of correlations
  - Hydrodynamic and thermal boundary layer
- Use of the integral analysis to calculate the heat transfer coefficient
- Natural convection
- Application of the Navier-Stokes equations to the convection problem

**Radiation**
- Use of the Stefan-Boltzmann Law, Planck's distribution Law and black body emissivity functions
- Emissivity, reflectivity and transmittance definitions and their use in spectral and gray surfaces
- Kirchoff's Law and multi-band width problems
- Definition and use of view factors
- Solution of multi-surface problems
- Combined modes of heat transfer problems

**Suggested References for Preparation:**


Problem 1 (20 Pts)
Consider Laguerre’s equation:
\[ xL_n'' + (1 - x)L_n' + nL_n = 0 \]

a) Derive the recursion relation for a series solution about x=0.

b) Show that when n is a non-negative integer, the recursion has a polynomial solution which terminates after a finite number of terms, and explicitly construct the first 3 of these polynomials, \( L_0, L_1, \) and \( L_2 \).

Problem 2 (40 Pts)
Consider the solid paraboloid of revolution described by:
\[ x^2 + y^2 \leq z \]
\[ 0 \leq z \leq 1 \]

Consider the vector field:
\[ \vec{v} = (x^3 + xy^2)\hat{i} + (y^3 + x^2y)\hat{j} \]
where, as usual, \( \hat{i} \) and \( \hat{j} \) are the unit vectors pointing along the x and y Cartesian axes.

a) Compute the integral over the boundary of \( \vec{v} \cdot \hat{n} \) where \( \hat{n} \) is the outward unit normal vector.

b) Compute the volume integral of the divergence of \( \vec{v} \) to explicitly verify the divergence theorem.

Problem 3 (40 Pts)
Consider a field \( u(r, \theta) \) governed by the Laplace equation, \( \nabla^2 u = 0 \). Recall that the form of the Laplace operator in polar co-ordinates is:
\[ \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \]

Solve for \( u(r, \theta) \) in a circular region of radius \( a \) subject to the following boundary condition:
\[ u(a, \theta) = T_1 \quad 0 \leq \theta < \pi \]
\[ u(a, \theta) = T_2 \quad \pi \leq \theta < 2\pi. \]
Problem 1 (10 Pts)
Suppose a matrix, $M$, is 3-by-3, real, and symmetric and has eigenvectors $\psi_1, \psi_2,$ and $\psi_3$ with corresponding eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Suppose the vector $x = 2\psi_1 - 2\psi_2$. If $M^4 x = a\psi_1 + b\psi_2 + c\psi_3$, what are $a, b,$ and $c$?

Problem 2 (20 Pts)
A circular disk of radius $a$ has a mass per unit area:
\[
\rho = \rho_0 \frac{y^2}{a^2 + y^2} e^{-(x^2+y^2)/\lambda^2}
\]
where the center of the circle is at $x = 0, y = 0$, $\lambda$ is a constant with units of length, and $\rho_0$ is a constant with units of mass per unit area. Calculate the total mass and leave your result in terms of $a, \rho_0,$ and $\lambda$.

Problem 3 (20 Pts)
A scalar field in 2 dimensions has the form:
\[
\phi(x, y) = \log[x^2 + y^2]
\]
What is the integral of the Laplacian, $\nabla^2 \phi = \nabla \cdot (\nabla \phi)$, over the region defined by $x^2 + y^2 \leq 1$? (Hint: you should use Gauss’s divergence theorem.)

Problem 4. (50 Pts): The temperature in a disk of radius $a = 5$ cm is governed by the diffusion equation
\[
\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T
\]
It is initially at a spatially uniform temperature of 100 K, and its edge is then quenched to a thermal bath fixed at $T = 0$ K. Solve for temperature distribution in the disk, $T(x, y, t)$. The thermal diffusivity of the disk is $\alpha^2 = 10^{-4}$ m$^2$/s. Assume the top and bottom faces of the disk are insulated. **Note: It is acceptable to express the answer in terms known functions or transforms.**
Appendix

The Diffusion Equation

The governing equation for thermal diffusion is of the form

\[ \alpha^2 \nabla^2 T = \frac{\partial T}{\partial t}, \]  

where \( T \) is the temperature, with

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  

\[ \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \]  

\[ \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial^2}{\partial \phi} \right) \]

in the cartesian, cylindrical and spherical coordinates, respectively.

Some useful second order ordinary differential equations

Euler-Cauchy Equations

An Euler-Cauchy equation is of the form

\[ x^2 y'' + bxy' + cy = 0 \]  

where \( b \) and \( c \) are constant numbers. These equations can be solved by using the change of variable \( x = e^t \).

Legendre Equation

The Legendre differential equation is the second-order ordinary differential equation

\[ (1 - x^2)y'' - 2xy' + l(l + 1)y = 0. \]  

The equation has two linearly independent solutions, a solution \( P_l(x) \) which is regular at finite points is called a Legendre function of the first kind, while a solution \( Q_l(x) \) which is singular at \( \pm 1 \) called a Legendre function of the second kind. If \( l \) is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial. The first few Legendre polynomials
are

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \]
\[ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \]
\[ P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) . \]

If the variable \( x \) is replaced by \( \cos \theta \), then the Legendre differential equation becomes

\[
\frac{d^2 y}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{dy}{d\theta} + l(l + 1)y = 0.
\] (7)

**Bessel Equation**

The differential equation

\[
x^2y'' + xy' + (x^2 - n^2)y = 0
\] (8)

has two classes of solution, called the Bessel function of the first kind \( J_n(x) \) and Bessel function of the second kind \( Y_n(x) \). It follows then \( J_n(kx) \) and \( Y_n(kx) \) are solutions to the equation

\[
x^2y'' + xy' + (k^2x^2 - n^2)y = 0
\] (9)

Some useful Bessel identities are as follows:

\[
\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x),
\]
\[
\int_0^c [J_n(kx)]^2x dx = \frac{c^2}{2}[J_{n+1}(kc)]^2
\]

**Laplace Transform**

The Laplace transform of a function \( f(t) \), defined for all real numbers \( t \geq 0 \), is the function \( F(s) \), defined by

\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt
\]