Decision Support

Maximizing revenue of end of life items in retail stores

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A B S T R A C T

Classical and even contemporary global supply chains have been built to reduce lead times and minimize costs. That has led companies to compete on the value of their supply chains as a competitive weapon. In this new era, many enterprises are realizing that supply chains can be revenue producing and help their sustainability efforts.

In this light, we consider the discount pricing of products that are being phased-out in retail stores. The decision to replace an item triggers a time horizon for the retailer to sell the existing inventory, hence the need to develop markdown strategies to deal with the issues of how much to discount the item, when to introduce these discounts, and if inventory should be left at the end of the time horizon to sell at some salvage price to a third party wholesaler. In this paper, we develop two models. The first one is an optimal markdown strategy that maximizes revenue from the discontinued items using multi-period nonlinear programming. The mathematical properties of the model are established and a closed form optimal solution is found. The second one is a linear programming model that addresses the issues of when and for how long during the phase-out time horizon to apply prices chosen from a pre-determined price set. These models are tested with real data provided by a retailer.

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1. Introduction

Inventory management is a crucial aspect of the bottom line regardless of what the business is. From accounting and financial perspective, inventory is an asset for businesses. Inventory is a buffer against uncertainty. However, inventory items have a limited life cycle. There is a window of opportunity to sell the product. Once that window closes then the sales value of it decreases and the profitability and inventory yield are not maximized. In addition to too much capital tied up, excess inventory influences service and operations. There are many reasons for excess inventory, and more often than none, the inventory buildup is not due to one reason only. Instead, multiple causes create the over-inventory situation. The multiple reasons reflect the lack of underlying priority, process and control.

In this paper, we will be addressing the excess inventory of products at the end of their life cycle, and determine the best pricing strategies for these phased-out items. Once the decision has been made to discontinue an item, the question becomes: when and by how much to markdown the item in order to deplete the given initial inventory on time. Making wrong decisions could result in a surplus of unnecessary inventory or worse, a loss in profits. In addition, we will also address the case of predetermined markdown prices, i.e. when and for how long to discount at each one of these prices. Fig. 1 below depicts the phase-out process, with an initial inventory ($I$) to be depleted during the phase-out time horizon ($T$), and either no inventory or some remaining inventory ($r$) at the end of the fixed phase-out time horizon is left to be sold to a third party wholesaler.

This paper was motivated by a real world problem facing a retailer, owning over 6000 stores in the U.S. A manufacturer/distributor, who supplies most of the items to the retailer, reviews all items and twice a year selects the ones to be phased-out. To retain anonymity, we will refer to them in this paper as retailer X and distributor Y. The phase-out time horizon is normally nine weeks. The items chosen for this research were taken from the cosmetic products category, which presented some of the most difficult inventory issues. The phase-out decision can be either a soft one, where there is only a change in packaging, or a hard one where the item is replaced by a different item. The replacement decision taken by the distributor can be either due to the non-sales or the short life cycle of certain products. Ineffective markdown strategies and sometimes failure to stop supplying

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the discontinued items during the phase-out time horizon, led to about dollar 10 million of total surplus inventory for retailer $X$.

In order to address the key problem at hand, historical sales data of the phased-out items were analyzed and a nonlinear regression model was developed to study the demand-price relationship. The model was then used to predict the behavior of an obsolete inventory item over a certain phase-out time horizon. Given an initial inventory to be depleted, the best price for each period can be determined in order to maximize revenue. In the case that the whole initial inventory is unable to be entirely depleted, and the option to sell the remaining inventory to a third party wholesaler at a particular salvage price is present, the model is modified to determine that remaining amount of inventory, and the new optimal prices for each period in order to maximize revenue.

The rest of this paper is organized as follows. Literature review is provided in Section 2. In Section 3, the price elasticity of demand and the demand-price relationship is derived from historical data followed by the development of two optimization models that maximize revenue from the discontinued items. Section 4 contains computational results and Section 5 conclusions and recommendations for future research.

Most often markdown prices in retail stores are not arbitrary but are selected from a set of predetermined prices. For example, if the regular price of an item is dollar 9.99, likely discount prices are dollar 8.99, dollar 7.99 and dollar 6.99. A second model is presented in this paper to determine the number of periods at which each predetermined markdown price is set during the phase-out time horizon in order to maximize total revenue from the phased-out items.

2. Literature review

Inelastic demand allows a producer to raise prices without much affecting demand for the product, whereas elastic demand exists when consumers are sensitive to the price at which a product is sold and will not buy if the price rises by what they consider excessive.

There has been an increasing adoption of dynamic pricing strategies and their further development in retail and other industries (Coy, 2000). Three factors contributed to this phenomenon: an increased availability of demand data, an ease of changing prices due to new technologies, and an availability of decision-support tools for analyzing demand data and for dynamic pricing.

Companies must be aware of their own operating costs and availability of supply, and they must have a good understanding of the customer's reservation price as well as the projection of future demand. Past research tried to address inventory problems, such as Whitin (1955) who was one of the first to highlight the fundamental connection between price theory and inventory control; Scarf (1960) who addressed optimal policies for multi-echelon inventory; and Porteus (1971) who examined a standard inventory model with a concave increasing ordering cost function. However, today, new technologies allow retailers to collect information not only about the sales, but also about demographic data and customer preferences (Elmaghraby & Keskinocak, 2003). Despite significant improvements in reducing supply chain costs via improved inventory management, a large portion of retailers still lose millions annually as a result of lost sales and excess inventory. Pourakbar, Frenk, and Dekker (2010) addressed the end of life inventory decisions for electronics service parts, and adopted an alternative policy to meet demands for service by offering customers a replacement of the defective product with a new one or giving a discount on the next generation of the product. Furthermore, Pourakbar, Van Der Laan, and Dekker (2012) presented a finite horizon Markov decision process to characterize the structure of the optimal inventory control policy. Continuing the work, Pourakbar, Van Der Laan, and Dekker (2014) addressed the returns of phase-out items, where a non-stationary demand arrival process is assumed. The structure of the optimal policy in the final phase is characterized and is shown that the optimal policy is a time-varying decision.

Other worthy literature in the inventory planning of old and new generations of the product together with the timing of the release of the product is mentioned in Li, Graves, and Rosenfield (2010). Furthermore, worthy literature that deals with service parts are mentioned by Hong, Koo, Lee, and Ahn (2008), Inderfurth and Kleber (2013), and Kleber, Schulz, and Voigt (2012).

According to Elmaghraby and Keskinocak (2003), there are three main characteristics of a market environment that influence the type of dynamic pricing problem a retailer faces: replenishment vs. no replenishment of inventory (R/NR), dependent vs. independent demand over time (D/I), and myopic vs. strategic customers (M/S). Based on different combinations of the 3 above-mentioned characteristics, different categories can be formed. Analytical models by Zhao and Zheng (2000) study how pricing decisions should be made in NR-I markets have the following common assumptions: the firm operates in a market with imperfect competition, the selling horizon $T$ is finite, the firm has a finite stock of $n$ items and no replenishment option, investment made in inventory is sunk cost, demand decreases in price $P$, and unsold items have a salvage value. Pricing decisions in such markets are mainly influenced by demand, and how it changes when prices change along with other factors (elasticity of demand). In particular, pricing decisions need to look at the arrival process of customers and the changes in the customer's willingness to pay over time.

Caro and Gallien (2010) presented a cost driven model where lacking service is translated into backorders and penalties, by using generic LTB and control policies. Feng and Chen (2003) considered a joint pricing and inventory control problem with setup costs and uncertain demand. Specifically, they developed an infinite horizon model that integrated pricing and inventory replenishment in a distribution environment, where they allowed for dynamically varying prices in response to changes in inventory levels by taking advantage of price-sensitive demand. In contrast, Zhao and Zheng (2000) modeled the demand as a non-homogenous Poisson process with rate $\lambda_t$ and allowed the probability distribution of the reservation price $F_t(P)$ to change over time.

Polatoglou and Sahin (2000) studied a periodic-review inventory model where, in addition to the procurement quantity, price is also a decision variable. They developed a model where demand in each period is a random variable having a price and, possibly, period-dependent probability distribution, with the expected demand decreasing in price.

When dealing with inventory replenishment, an eye must be kept on the effects of setting the price too low or too high. If it is too low, it could risk stock-outs and lost sales while waiting for replenishment. And if set too high, it could lead to excess inventory and high holding costs.

Two interesting approaches to the end of life products were addressed by Caro and Martinez-de-Albeniz (2010) who looked at the impact of quick response in inventory-based competition by considering two different retailers selling a substitutable product over a finite horizon that is divided into two periods. Another paper by Leifker, Jones, and Lowe (2012) addressed the end-of-life parts acquisition with limited customer information by examining different types of replenishment scenarios.

The clearance period is defined as the period bounded by the first markdown and the “outdate” when all remaining inventory is salvaged and new items arrive to replace the old ones on the store shelves (Zhao & Zheng, 2000). Caro and Gallien (2012) designed and implemented an alternative process to Zara’s current decision-making process relying on a formal forecasting model. Gupta, Hill, and Bouzidine-Chameeva (2004) proposed discrete-time models to deal with the problem of setting prices for clearing retail inventories of fashion goods. Recent work to address modeling sourcing strategies to mitigate part obsolescence was published by Shen and Willems (2014). However, most of the research mentioned above does not explicitly address the multiple factors that affect pricing decisions of phased-out items in retail stores. Furthermore, most of the research performed considers a single store, because the problem becomes far more complicated for retail chains especially if they need to coordinate their inventories and prices across all stores. In this paper, factors such as price elasticity of demand, salvage price, initial inventory level and limited phase-out horizon are simultaneously considered to find the optimal pricing decisions for the phased-out items.

3. Model development and solution procedure

Markdowns near the end of the phase-out time horizon have less of an impact on depleting the inventory. The main focus as time passes and moves closer toward the end of the phase-out time horizon is: timing the markdown, and sizing the discount at every decision period. To address the two issues properly, two models are presented in this paper. The first model seeks an optimal markdown strategy using multi-period nonlinear programming that maximizes revenue from the discontinued items. The second one is a linear programming model that addresses the issues of when and for how long to apply pre-determined markdown prices during the phase-out time horizon. The models are tested with real data provided by a retailer.

3.1. Price elasticity of demand

To predict consumer’s buying behavior, techniques have been developed to evaluate consumers’ sensitivity to changes in price; the most commonly used measure is the “price elasticity of demand.” Elasticity of demand is the ratio of the percentage of the change in demand (ΔQ) with respect to the percentage of the change in price (ΔP): 

\[ E_d = \frac{\Delta Q}{\Delta P} \]

In the case that products do not sell well, retailers tend to use aggressive markdown strategies to try to salvage and maximize the return of whatever is left in the inventory.

In this research, a large number of phased-out inventory items (SKUs) in a wide number of retail stores (customers) were analyzed for a combined 54-week period. Assumptions of independence between the sales of the individual SKUs were made. Since unit price was not readily available, “Total Sales Amount (TSA) in dollar and Total Sales Volume (TSV) in units” were used to calculate it, by dividing TSA by TSV.

Using the data, several regression functions of the weekly sales volume (V) versus selling price (P) were tested; among them, a linear, a quadratic and a power function. For all the SKUs under study, the power regression function \( V = aP^\beta \) yielded the best fit, where \( \alpha \) and \( \beta \) are the parameters of the function. The power function is used very often in econometrics to express the relationship between demand and price. Parameter \( \beta \) is called price elasticity of demand. The price elasticity is always \( \beta \leq 0 \), while \( \alpha \) is a positive constant. Nonlinear regression analysis provides the two parameters as follows, where \( P_j \) and \( V_j \) are the price and sales volume in week \( j = 1, \ldots, n \), respectively.

\[
\begin{align*}
\ln(\alpha) &= \frac{\sum_{j=1}^{n} \ln(V_j) - \beta \sum_{j=1}^{n} \ln(P_j)}{n} \\
\beta &= \frac{n \sum_{j=1}^{n} \ln(P_j) \ln(V_j) - \sum_{j=1}^{n} \ln(P_j) \sum_{j=1}^{n} \ln(V_j)}{n \left( \sum_{j=1}^{n} \ln(P_j)^2 \right) - \left( \sum_{j=1}^{n} \ln(P_j) \right)^2}
\end{align*}
\]

As it is explained later in Section 4, cluster analysis was necessary to be performed on the data points \( (P_j, V_j) \) before they could be used to estimate parameters \( \alpha \) and \( \beta \). The following table provides an interpretation of the values of the price elasticity coefficient \( \beta \) (Table 1).

<table>
<thead>
<tr>
<th>Value</th>
<th>Descriptive terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta = 0 )</td>
<td>Perfectly inelastic demand</td>
</tr>
<tr>
<td>(-1 &lt; \beta &lt; 0 )</td>
<td>Inelastic or relatively inelastic demand</td>
</tr>
<tr>
<td>( \beta = -1 )</td>
<td>Unitarily elastic demand</td>
</tr>
<tr>
<td>( -\infty &lt; \beta &lt; -1 )</td>
<td>Elastic or relatively elastic demand</td>
</tr>
<tr>
<td>( \beta = -\infty )</td>
<td>Perfectly elastic demand</td>
</tr>
</tbody>
</table>

![Fig. 2. Effect of different values of \( \beta \) on sales volume.](image-url)
and quantity demanded. This behavior is evidently depicted in Fig. 2, where the effects of $\beta$ are clearly shown. In the remainder of this paper the power sales volume function $V = \alpha \beta$ with $\beta < -1$ is assumed to describe the deterministic relationship between the sales volume ($V$) and the selling price of the item ($P$).

3.2. A revenue maximization model for phased-out items

Since the SKUs are considered independent, i.e. sales volume of one does not affect the sales volume of another, one can optimize the revenue from each individual SKU separately, thus considering a single-item system. The revenue ($R$) is determined by multiplying the unit price of the item by the sales volume function, or $R = V \cdot P = \alpha \beta$. The major optimization problem that is proposed here is to determine the unit price $P_t$ to charge each SKU ($i$) in each period $t$, so that the resulting total revenue is maximized. The single SKU and multiple periods’ revenue optimization problem that determines the optimal price $P_t$ at each period $t$ can be formulated as follows:

Maximize $\sum_{t=1}^{T} \alpha \beta^{t-1} P_t$ 

Subject to: $\sum_{t=1}^{T} \alpha \beta^{t-1} P_t \leq I$ 

$R_t \geq 0$, $t = 1, \ldots, T$ 

where $I$ is the initial inventory to be depleted and $T$ is the total number of periods during the phase-out process. Objective function (1.1) maximizes total revenue from the single SKU during phase-out time horizon $T$. Constraint (1.2) assures that total volume sold during that time does not exceed the initial inventory $I$, and constraint (1.3) restricts prices to non-negative numbers.

A more general model results when remaining (unsold) inventory is allowed to be sold at the end of $T$ periods. One option is to try to sell the remaining inventory to a third party wholesaler. Let $r$ be the remaining inventory and $C$ be its salvage price, respectively. The additional revenue associated with selling this residual inventory at the salvage price is now added to the objective function and the residual inventory $r$ is added to the left hand side of the inventory constraint. The mathematical formulation of the more general model with residual inventory becomes:

Maximize $\sum_{t=1}^{T} \alpha \beta^{t-1} P_t + Cr$ 

Subject to: $\sum_{t=1}^{T} \alpha \beta^{t-1} P_t + r \leq I$ 

$R_t \geq 0$, $t = 1, \ldots, T$ 

$r \geq 0$ 

It should be noted that when residual inventory is not allowed, i.e. $C = 0$, the value of $r$ in the optimal solution will be 0 and thus the general model reduces to the above model. Also, when residual inventory is allowed, $C > 0$, as in the above model, (2.2) will always be satisfied as an equality.

3.2.1. Solution methodology

The above general problem described by (2.1)-(2.4) is a nonlinear optimization problem. Its objective function is convex as the sum of convex functions. Note that the second derivative of $\alpha \beta^{t-1}$ with respect to $P_t$ is $\alpha \beta (\beta + 1) \beta^{-1}$ which is positive for $\alpha > 0$ and $\beta < -1$. Similarly, since the summation terms $\alpha \beta^t$ of constraint (2.2) are convex, the left hand side of constraint (2.2) is a convex function, and the feasible region of the problem defined by constraints (2.2)-(2.4) is convex. The optimal solution of such an optimization problem, where the maximizing objective function is convex and the feasible region is convex, is at the boundary of the feasible region. To solve this optimization problem we apply the necessary Karush-Kuhn-Tucker (KKT) conditions for optimality as follows.

Let $u$, $v_t$ ($t = 1, \ldots, T$) and $w$ be the Lagrange multipliers associated with constraints (2.2), (2.3) and (2.4), respectively. The KKT necessary conditions for optimality at ($P_1, \ldots, P_T, r$) are:

$\alpha(\beta + 1)P_t - u \alpha \beta P_t^{\beta - 1} + v_t = 0$, $t = 1, \ldots, T$ 

$C - u + w = 0$ 

$u \geq 0$ 

$w \geq 0$ 

$v_t \geq 0$, $t = 1, \ldots, T$ 

$u \left( \sum_{t=1}^{T} \alpha \beta^t + r - I \right) = 0$ 

$r \leq 0$ 

$C = 0$, $w = 0$ 

$\sum_{t=1}^{T} \alpha \beta^t + r \leq I$ 

$P_t \geq 0$, $t = 1, \ldots, T$ 

$r \geq 0$ 

Eqs. (2.5) and (2.6) are the optimality conditions, Eqs. (2.7)-(2.9) assure that all Lagrange multipliers are non-negative, (2.10)-(2.12) are the complementary slackness conditions, and constraints (2.13)-(2.15) are the feasibility conditions.

Observing (2.6), let us assume that $u = 0$. Then $w = -C < 0$, which contradicts the non-negativity of $w$. Therefore, $u > 0$ and (2.10) implies that

$\sum_{t=1}^{T} \alpha \beta^t + r = I$ 

Furthermore, since $\lim_{P_t \to 0} \alpha \beta^t = \infty$, constraint (2.13) is violated for $P_t = 0$, $t = 1, \ldots, T$. Therefore, $P_t > 0$, $t = 1, \ldots, T$, which implies from (2.11) that $v_t = 0$, $t = 1, \ldots, T$ and (2.5) yields $R_t = \left( \frac{C}{\beta + 1} \right) u$, $t = 1, \ldots, T$.

Finally, $r$ can be positive or zero. If $r > 0$, then (2.12) implies that $w = 0$, and (2.6) becomes $u = C$. Therefore, $P_t = \left( \frac{C}{\beta + 1} \right) C$, $t = 1, \ldots, T$.

In addition, when $r = 0$, (2.16) yields $R_t = \left( \frac{C}{\beta + 1} \right) t^{1/\beta}$, and $u$ and $v_t$ become:

$u = \left( \frac{C}{\beta + 1} \right) P_t = \left( \frac{C}{\beta + 1} \right) \left( \frac{I}{\beta} \right)^{1/\beta}$

$\beta + 1$ 

In addition, when $r = 0$, (2.16) implies that $w \geq 0$, and combining that with (2.6), $w = u - C \geq 0$ is obtained. Substituting $u$ from above, results in the following inequality for $C$:

$C \leq \left( \frac{\beta + 1}{\beta} \right) \left( \frac{I}{\beta} \right)^{1/\beta}$ 

Let

$C_0 = \left( \frac{\beta + 1}{\beta} \right) \left( \frac{I}{\beta} \right)^{1/\beta}$ 

(2.17)
be the threshold salvage price at or below which residual inventory (r) should not be kept. The results of the above KKT conditions are summarized in the following theorem and corollary.

**Theorem 1.** When the salvage price exceeds a threshold price C₀, the optimal policy in the revenue maximization problem is to set the same price in all periods during the phase-out time horizon \( P₁ = (\frac{1}{\beta r})C₁, \) \( i = 1, ..., T \) and the residual inventory will be \( r = l - Ta(\frac{\beta}{\beta + 1}). \)

**Corollary 1.** If the salvage price is at or below the threshold price C₀, the optimal policy is to leave no inventory at the end of the horizon \( r = 0 \), and to set the price in all periods at \( P_0 = (\frac{1}{\beta r})C₀, t = 1, ..., T. \)

The above theorem and corollary provide an elegant closed form solution to the revenue maximization problem for phased-out items. The formulas are expressed in terms of the model parameters. In a graphical form, \( P₁ \) is depicted in Fig. 3 as a function of the salvage price C. \( P₁ \) is constant at \( (\frac{1}{\beta r})C₁ \) for all salvage prices less than or equal to \( C₀ \) and proportional to \( C₀ \) for salvage prices greater than \( C₀ \), at a slope \( \beta(\frac{1}{\beta + 1}). \)

Since \( \beta < 1, (\frac{1}{\beta r})C₀ \) can be written as \( (\frac{\alpha}{\beta r})^{1/\beta} \), where \( |\beta| > 1 \) is the absolute value of the price elasticity parameter \( \beta \) and \( lT \) is the average inventory per period that needs to be depleted. Therefore, the optimal price \( P₁ \) for \( C < C₀, P₁ \) is increasing when \( lT \) is decreasing. This is intuitive since the lower the initial inventory is and the longer the time to sell it, the higher the assurance that the inventory is sold before the end of the phase-out time horizon and therefore the higher the price can be set. In addition, the more elastic the demand is, the lower \( lT|\beta| \) is and therefore the lower the price \( P₁ \) becomes. An interpretation of this is that in order to maintain certain demand the price should be lowered when demand becomes more elastic (see Fig. 2). The threshold price \( C₀ \) can be very useful to a retailer when implementing a pricing policy for a phased-out item. If the salvage price is below \( C₀ \), it is more profitable to sell the whole inventory before reaching the end of the phase-out horizon. Also note that \( C₀ \) can be expressed similarly as \( C₀ = (\frac{1}{\beta r})(\frac{\alpha}{\beta})^{1/\beta}. \) This implies that \( C₀ \) is decreasing with the average inventory that needs to be sold per period \( (lT) \). The intuitive interpretation is that lower salvage prices are tolerated as the average inventory per period \( (lT) \) increases.

When the above optimal policy is implemented, the actual volume sold in a period \( t \) may deviate from the model’s prediction, \( V₁ = \alpha P₁^{\beta} = (\frac{\beta}{\beta + 1})^{\beta}. \) This suggests a dynamic pricing strategy where the remaining inventory \( l \) and remaining periods \( T \) are updated after each period and the price for the next period is recomputed by the model.

### 3.3. A revenue maximization model with pre-determined prices

Most often retail chains choose prices from a pre-determined price set. Market competition and psychological reasons may dictate such a price strategy. Consider a set of pre-determined prices \( (D₁, D₂, ..., D_n) \) ordered in decreasing values, i.e. \( D₁ > D₂ > ... > Dₙ, D₁ \) may be the current price (before any discount) and \( Dₙ \) should be higher than the salvage price \( C, Dₙ > C. \) The volume of units sold, \( V₁, \) of the item (SKU) under consideration at price \( Dᵢ \) is \( Vᵢ = \alpha Dᵢ^{\beta} \) and the associated revenue is \( Rᵢ = \alpha Dᵢ^{\beta + 1}, \) \( i = 1, ..., n. \) Since \( \alpha > 0 \) and \( \beta < 1, \) \( V₁ < V₂ < ... < Vₙ \) and \( R₁ < R₂ < ... < Rₙ. \) The revenue maximization model with pre-determined prices below determines the number of periods \( xᵢ, \) a phased-out item (SKU) is set at price \( Dᵢ, i = 1, ..., n, \) during the phase-out horizon.

Maximize \( \sum_{i=1}^{n} Rᵢxᵢ + Cr \) \hspace{1cm} (3.1)

Subject to: \( \sum_{i=1}^{n} Vᵢxᵢ + r ≤ I \) \hspace{1cm} (3.2)

\( \sum_{i=1}^{n} xᵢ ≤ T \) \hspace{1cm} (3.3)

\( r ≥ 0 \) \hspace{1cm} (3.4)

Objective function (3.1) and constraint (3.2) are similar to (2.1) and (2.2), respectively, except that volume sold and revenue collected in a time period \( (Vᵢ \) and \( Rᵢ), \) are known parameters for a given price \( Dᵢ. \) Constraint (3.3) ensures that the total number of periods where we use pre-determined prices does not exceed the number of periods in the phase-out time horizon \( T. \) Constraint (3.4) restricts the values of decision variables \( xᵢ \) to non-negative values. If \( xᵢ \) is only acceptable in integer form, then after the problem is solved the values of \( xᵢ \) can be rounded to the closest integer for an approximate optimal solution, or alternatively the problem can be solved by an integer programming software.

The problem defined by Eqs. (3.1)–(3.5) above is a linear optimization problem that can be solved by the simplex algorithm. However, looking closely at the problem, one can notice the structure of the Linear Multiple Choice Knapsack (LMCK) problem, which can be solved by a very simple if-then-else structure. In addition, casting the problem as a LMCK problem instead of an integer programming problem provides further insights such as the concept of the incremental rate of revenue between two prices, its relation to the salvage value, and the importance of the average inventory to be depleted per period, \( lT, \) in determining the optimal solution.

The LMCK problem can be viewed as the problem of maximizing the total value of objects inserted in a knapsack of capacity \( l \) with the additional stipulation that the objects are partitioned into disjoint subsets and a limited quantity of the objects in each subset should be selected. In our problem, the first subset comprises the prices \( (1, ..., n) \) on which the decision variables \( xᵢ, i = 1, ..., n, \) are defined. There is a limited number of periods \( (T) \) in which the prices \( Dᵢ \) can be set, i.e. \( \sum_{i=1}^{n} xᵢ ≤ T \) is the multiple-choice constraint of the first subset. At each price \( Dᵢ, \) volume \( Vᵢ \) is sold and revenue \( Rᵢ \) is collected per period. The inventory volume \( Vᵢ \) plays the role of the unit weight of object \( i \), and the total inventory \( I \) to be depleted plays the role of the knapsack’s capacity. The second subset contains a single item, the residual inventory at the end of the phase-out time horizon. Its unit weight is 1 (coefficient of \( r \) in constraint (3.2)) and its unit revenue is \( C. \) The upper bound on the quantity of this item is the initial inventory \( I \). There is no need for
an additional multiple-choice constraint (r ≤ I) because (3.2) takes care of that. The sets of decision variables associated with the two multiple-choice sets are \(x_1, x_2, ..., x_6\) and \(r\).

The LMCK problem has been widely investigated (Lin, 1998). It has found applications in sales resource allocation and menu planning (Lin, 1998) and in allocating funds to highway safety improvements (Melachrinoudis & Kozanidis, 2002), among others. Most of the solution algorithms are based on the fact that, any decision variable of the problem that satisfies either of two special conditions, can be eliminated from further consideration (Sinha & Zolt­ners, 1979) as dominated variable. Thus the size of the problem can be significantly reduced. The elimination of dominated variables in a subset can be graphically illustrated by depicting each element of the subset as a point on the plane with coordinates being its unit weight and its unit revenue. In the prices subset, price \(i\) is represented as point \((V_i, R_i)\). Note that the slope of the point is the benefit-cost ratio, being in this case the price itself, \(D_i = \frac{R_i}{V_i}\). The upper left hull of points \((V_i, R_i)\) determines the non-dominated frontier which is a piecewise linear and concave function. Every variable \(x_i\) associated with a point \((V_i, D_i)\) below the upper left hull is dominated and therefore can be eliminated from further consideration. In the plot of the second subset there is only one point \((1, C)\), its slope is \(C\) and the upper left hull is the line segment connecting the origin with point \((1, C)\). There is only one variable in this subset \((r)\) which is non-dominated. For the prices subset, consider plotting points in decreasing order of prices \(D_i\), or equivalently in increasing order of unit volume and revenue, i.e., \((V_1, R_1), (V_2, R_2), ..., (V_n, R_n)\). The first point has the highest slope \((D_1)\). Connect in the same order \((V_1, R_1)\) with \((V_2, R_2)\) with \((V_3, R_3)\) and so on. The slope of the line segment connecting two sequential points \((V_i, R_i)\) and \((V_{i+1}, R_{i+1})\) is the incremental rate of revenue, \(s_i = \frac{R_{i+1} - R_i}{V_{i+1} - V_i}\), i.e. the rate of increase in revenue per unit volume sold when we switch from price \(D_i\) to \(D_{i+1}\). Consider \(S_0 = S_1 = D_1\) by setting \(R_0 = V_0 = 0\). The following lemma asserts that the incremental rate of revenue is non-negative and is decreasing with \(i\). The first fact is intuitive but the second is not. As we switch from price \(D_1\) to \(D_{i+1}\), the price is lower, the revenue is higher because we sell more.

**Lemma 1.** Consider three sequential points \((V_i, R_i)\), \((V_{i+1}, R_{i+1})\), \((V_{i+2}, R_{i+2})\), associated with prices \(D_i \geq D_{i+1} \geq D_{i+2}\), where \(Y_i = \alpha D_i^\beta\) and \(R_i = \alpha D_i^\beta + 1\), \(i = 1, ..., n, \) then the incremental rate of return is decreasing with \(i\), or \(R_{i+2} - R_{i+1} \geq R_{i+1} - R_i \geq 0\), \(i = 0, ..., n-2\).

**Proof.** Since \(\beta < -1 \Rightarrow \beta + 1 < 0\) and \(\beta + 1 > 0\). Furthermore, \(\beta + 1 > \beta \) implies that \(Y_i = \alpha D_i^\beta\), therefore, for \(\beta < -1\), \(0 < \gamma < 1\). Consider the transformation \(Y_i = \alpha D_i^\beta\), \(i = 1, ..., n\). Since \(D_i \geq D_{i+1} \geq D_{i+2}\), \(Y_i \leq Y_{i+1} \leq Y_{i+2}\), substituting \(\gamma\) for \(\beta\) and using the transformation, the inequality \(\frac{R_{i+2} - R_{i+1}}{V_{i+2} - V_{i+1}} \geq \frac{R_{i+1} - R_i}{V_{i+1} - V_i} \geq 0\) becomes \(\frac{Y_{i+1} - Y_i}{V_{i+1} - V_i} \geq \frac{Y_i - Y_{i-1}}{V_i - V_{i-1}} \geq 0\), which holds because \(Y^\gamma\) is a concave function of \(Y\) (\(\gamma\) is a constant and \(0 < \gamma < 1\)) □.

The significance of the above lemma is that all points \((V_i, R_i)\) in the multiple-choice subset are used to define the upper left hull of points, which in turn implies that no price is dominated. The upper left hull is illustrated in Fig. 4 where points \((V_i, R_i)\) for five pre-determined prices of a SKU have been plotted and connected sequentially by line segments forming a piecewise linear and concave curve. None of the points \((V_i, R_i)\) is below the upper left hull, i.e., none of the prices should be eliminated.

The algorithm by Johnson and Padberg (1981) for the LMCK problem orders in decreasing value the slopes of points of each subset in a list and then it merges all lists in a master list. For the sake of illustration, let us assume that the value of the salvage price \(C\) is between the second and third slope of the prices subset, or the master list contains the slopes \(D_1, D_2, D_3, D_4, D_5\) and \(C\). The list of variables associated with the slopes is \((x_1, x_2, r, x_3, x_4, x_5)\). The algorithm starts with the first variable in the list by increasing it to the maximum possible amount (maximum value of \(x_i\) is \(T\) and maximum value of \(r\) is \(I\)). Increasing a variable implies that some amount of the resource (inventory) is used \((X_i/r\) for price \(i\) and \(r\) for the residual inventory) and the remaining amount of resource needs to be updated. If there is resource left, the algorithm proceeds with the next variable in the list by increasing it as much as possible and decreasing the previous positive variable. This step continues until either all the resource \((I)\) is depleted or all variables in the list have been considered. The algorithm terminates with no more than two variables in each subset having positive values. The two variables are adjacent in the variable list. The adaptation of Johnson and Padberg’s (1981) algorithm for the revenue maximization problem with pre-determined prices is as follows:

**Algorithm 1.**

*Input: \(V_i, R_i, i = 1, ..., n\) and \(V_0 = R_0 = 0\), \(I, T, C\).

Step 1. Compute \(s_{i+1} = \frac{R_{i+1} - R_i}{V_{i+1} - V_i}, i = 0, ..., n-1\).

Let \(k = 0\), if \(V_n > \frac{T}{I}\), and \(k = \max\{i | V_i \leq \frac{T}{I}\}\) otherwise.

Let \(j = \max\{i | s_i \geq C\} > 0\).

Step 2.

Step 2.1. If \(k = 0\), set \(x_i = \frac{I - V_k}{T - V_k} x_i = 0, i \neq k, r = 0\).

Step 2.2. else if \(1 < j \leq k\), set \(x_j = \frac{I - V_k}{T - V_k} x_i = 0, i \neq j, r = I - T - V_k\).

Step 2.3. else if \(j < k\), set \(x_j = \frac{I - V_k}{T - V_k} x_i = 0, i \neq k, x_k = 0, i \neq k, r = 1\).

End if.

*Output: \(x_i, i = 1, ..., n\).

The maximum number of variables that can be positive in the above algorithm is two. This is expected since the linear program has two functional constraints. The algorithm provides the following intuitive interpretation. When the initial inventory is very low, \(I < V_i/T\), there is no need to discount; the inventory will be depleted before the end of the phase-out time horizon even if sold at the regular price \(D_1\), i.e. \(X_1 = \frac{I}{T} < T\) (Step 2.1). If the salvage price is attractive (high) while the average inventory \(I/T\) is high, the price is set at price \(D_1\) in all periods \((X_j = T\) where \(D_i\) is the price associated with the incremental rate of return which is immediately higher than \(C\); the remaining inventory is sold at the salvage price (Step 2.2). Finally, when both the salvage price and the average inventory are low, then no residual inventory is left and two successive \(x_k\) and \(x_{k+1}\) are positive, determined by \(V_k \leq \frac{T}{I} \leq V_{k+1}\) (Step 2.3). Note that in this case, \(x_k + x_{k+1} = T\).
4. Computational results

The model was tested with data of obsolete cosmetic items supplied by retailer X, which has more than 6000 stores all across the United States. The weekly total sales volume and the total sales amount were provided for a period of 54 weeks. 6 SKUs were randomly selected from a family of SKUs exhibiting price elasticity behavior, labeled SKU 111 through 116 for privacy reasons. For illustration, SKU number “116” is considered below.

There were very few weeks associated with low prices than weeks with higher prices. This was true for other SKUs as well. It was evident therefore that there was a need to cluster the data to determine a better understanding of the relationship between price and demand. Thus, a k-means clustering algorithm (Duda, Hart, & Stork, 2001) was used, where the number of clusters k had to be chosen first. Each cluster of points was replaced by a single point, its centroid. Using k = 9, the new points after clustering are listed by their coordinates (price and sales volume) in Table 2 and displayed in Fig. 5, respectively.

The same procedure was followed for all the SKUs under study. The next step was to find the best-fit model using regression analysis. As mentioned earlier, the power regression function yielded the best fit. The power equation and the coefficient of determination (R^2) for SKU #116 are shown on the top right of Fig. 5.

To compare the price elasticity of the SKUs under study, we normalized their power functions so that the (standardized) price is the fraction of the current price and the sales volume is the number of times the volume increases compared to the sales volume at the current price. Table 3 and Fig. 6 below show the results. Looking at Fig. 6, we see that when the price of the least elastic item (SKU 116) is lowered to 50 percent, its sales volume increases to about three times; for the most elastic item (SKU 114) the sales volume increases to about six times. The normalized power functions for the six SKUs are shown at the bottom of Table 3, where the price elasticity parameter (β) is in parentheses.

The demand-price power functions obtained for the six SKUs were used to test the two optimization models. Under the assumption that remaining inventory is not allowed at the end of the phase-out time horizon (T = 0), the first model provides the optimal discount price, same for every period, \( P = \left(\frac{1}{(1 - \beta)}\right)^{1/\beta} \). The optimal prices obtained together with the maximum revenue (optimal objective function value), \( R_{\text{max}} = \frac{1}{2} \left(\frac{\beta}{(1 - \beta)}\right)^{1/\beta} \), are displayed for the six SKUs in Table 4 for the corresponding initial inventory levels (I) and T = 9 weeks.

To show the proposed model’s advantage over the retailer’s current markdown strategy, we simulated the current practice on the same SKUs by calculating projected sales volumes V using discounted prices \( P \) and the adopted power function model \( V = \alpha P^\beta \). The retailer’s markdown strategy is as follows. The price is lowered by 25 percent in the first three weeks of the phase-out time horizon (9 weeks) and sales are observed. The markdown is considered effective if at least 1/3 of the initial inventory is depleted. In that case the price stays the same for the next three weeks and is reassessed at the end of the sixth week. If the markdown is not effective in the first three weeks, the retailer moves to the next markdown level (50 percent). The markdown process is repeated at the end of the sixth week. If the 50 percent markdown is not effective, the retailer moves to the last markdown level of 75 percent.

The results of the retailer’s strategy are displayed and compared to the results of the proposed model for the six SKUs in Table 5. The revenue improvement ranges from 8.6 percent to 22.6 percent for the six SKUs. With the exception of SKU #112, for which a 25 percent markdown throughout the phase-out time horizon worked in depleting all inventory, a markdown of 50 percent had to be applied after the third week for all inventories to be depleted. Note that if initial inventories were much higher, items would have been left unsold even with the strongest markdown schedule 25–50–75 with further deterioration in revenues. Our proposed model optimized revenue and depleted all inventory.

Next, we illustrate the proposed model when the salvage price is greater than certain threshold value \( C_0 \), given by (2.17). In that case, it is worth leaving inventory at the end of the phase-out time horizon. Table 6 shows the threshold salvage values for all SKUs. For example, for SKU #116, \( C_0 = 2.15 \). The optimal discount price of SKU #116 as a function of salvage price \( C \) has been plotted in Fig. 3. When the salvage price is less than or equal to dollar 2.15, the price is set to dollar 5.70 (see Table 4) and no inventory is left at the end of the phase-out time horizon. When the salvage
price is greater than dollar 2.15, the price of the item is increasing linearly with the salvage price (see Fig. 3). The threshold salvage price can be very important to a retailer when deciding to accept or reject the salvage price offered for a phased-out item by a third party wholesaler.

If the markdown prices are set in advance and they are fixed, the question becomes: when to use them and for how many periods during the phase-out time horizon.

Five pre-determined prices were assumed for SKU # 116: \( D_1 = 8, D_2 = 7, D_3 = 6, D_4 = 5 \) and \( D_5 = 4 \). The volume of units sold \( V_i \) and the revenue collected each week \( R_i \) is computed by \( V_i = \alpha D_i^p \) and \( R_i = \alpha D_i^{p+1} \) for each price \( i = 1, ..., 5 \) and is shown in Table 1. In addition, the incremental rate of revenue is computed by \( s_{i+1} = \frac{R_{i+1} - R_i}{V_{i+1} - V_i}, i = 0, ..., 4 \) and is included in Table 7. Note that \( V_0 \) and \( R_0 \) were defined as 0 earlier. The points \( (V_i, R_i) \) and the associated slopes \( (s_i) \) for SKU # 116 were illustrated earlier in Fig. 4. The two important factors that determine the solution in the second model are the average inventory per time period \( (I/T) \) ratio, and the salvage price \( (C) \). as explained previously.

For initial inventory \( I = 20,742 \) and a total phase-out time horizon of \( T = 9 \) weeks, the average inventory per time period is computed as \( I/T = 2305 \). Algorithm 1 is illustrated in Fig. 7 for several values of salvage price varying from dollar 0 to dollar 4.

Since \( V_0 < \frac{T}{I} < V_4 \), Step 1 of the algorithm yields \( k = 3 \). Let us compute the values of \( j \) for various hypothetical values of \( C: j = 1 \), if \( C = 3, 4; j = 2 \), if \( C = 2.5; j = 4 \), if \( C = 2.0 \); and \( j = 5 \), if \( C = 0, 0.5, 1.0, and 1.5 \). If \( C = 3, 4 \), then \( j = 1 \) and \( k = 3 \). Therefore, \( 1 \leq j \leq k \) and Step 2.2 of the algorithm is executed, yielding \( x_1 = 9; x_2 = 0 \) for \( i \neq 1 \); and \( r = 8712 \). If \( C = 2.5 \), then \( j = 2 \) and \( k = 3 \). Therefore, \( 1 \leq j \leq k \) and Step 2.2 of the algorithm is executed, yielding \( x_2 = 9; x_3 = 0 \) for \( i \neq 2 \); and \( r = 5837 \). If \( C = 0, 1.0, 1.5, 2.0 \), then \( 1 \leq j < k \). Therefore, Step 2.3 of the algorithm is executed, yielding \( x_3 = 6.71 \); \( x_4 = 2.29 \); \( x_5 = 0 \) for \( i \neq 3, 4 \); and \( r = 0 \). The \( x_i \) values may be rounded to the closest integers \( (x_3 = 7 \) and \( x_4 = 2.3) \).

Table 4

<table>
<thead>
<tr>
<th>SKU #</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>Initial inventory</th>
<th>Optimal price</th>
<th>Max revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>116</td>
<td>37,625</td>
<td>-1.605</td>
<td>20,742</td>
<td>5.70</td>
<td>118,237</td>
</tr>
<tr>
<td>115</td>
<td>69,700</td>
<td>-1.764</td>
<td>28,726</td>
<td>5.75</td>
<td>165,043</td>
</tr>
<tr>
<td>114</td>
<td>10,325</td>
<td>-2.408</td>
<td>32,005</td>
<td>4.05</td>
<td>12,975</td>
</tr>
<tr>
<td>113</td>
<td>16,946</td>
<td>-2.574</td>
<td>39,300</td>
<td>4.14</td>
<td>16,294</td>
</tr>
<tr>
<td>112</td>
<td>12,941</td>
<td>-2.493</td>
<td>31,888</td>
<td>4.24</td>
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</tr>
<tr>
<td>111</td>
<td>11,362</td>
<td>-2.323</td>
<td>40,583</td>
<td>4.01</td>
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Table 5

<table>
<thead>
<tr>
<th>SKU #</th>
<th>Current practice</th>
<th>Proposed model</th>
<th>Revenue improvement (percent)</th>
</tr>
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<tr>
<td></td>
<td>Markdown schedule</td>
<td>Total revenue (dollar)</td>
<td>Max revenue (dollar)</td>
</tr>
<tr>
<td>116</td>
<td>25-50-50</td>
<td>99,292</td>
<td>118,237</td>
</tr>
<tr>
<td>115</td>
<td>25-50-50</td>
<td>136,844</td>
<td>165,043</td>
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<tr>
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<td>13,711</td>
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Table 6

<table>
<thead>
<tr>
<th>SKU #</th>
<th>Minimum salvage price required</th>
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<tr>
<td>116</td>
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<tr>
<td>115</td>
<td>2.49</td>
</tr>
<tr>
<td>114</td>
<td>2.37</td>
</tr>
<tr>
<td>113</td>
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<td>112</td>
<td>2.54</td>
</tr>
<tr>
<td>111</td>
<td>2.28</td>
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Table 7

<table>
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<tr>
<th>( t )</th>
<th>Value of price ( (D_t) )</th>
<th>Volume ( (V_t) )</th>
<th>Revenue ( (R_t) )</th>
<th>Incremental rate of revenue ( (s_t) )</th>
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<tr>
<td>1</td>
<td>8</td>
<td>1337</td>
<td>10,696</td>
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<td>2</td>
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<td>1636</td>
<td>11,592</td>
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<td>2942</td>
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<td>5</td>
<td>4</td>
<td>4066</td>
<td>16,264</td>
<td>1.68</td>
</tr>
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</table>

Fig. 6. Normalized power functions of the six different SKUs.
(3) the case of stochastic demand, and (4) developing an efficient algorithm for the revenue maximization problem with predetermined prices, in which the variables are general integers by casting the problem into the class of Multiple Choice Knapsack problems.

References


### Table 8

<table>
<thead>
<tr>
<th>C</th>
<th>x1</th>
<th>x2</th>
<th>x3</th>
<th>x4</th>
<th>x5</th>
<th>Remaining inventory</th>
<th>Maximum revenue</th>
</tr>
</thead>
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<tr>
<td>4.00</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8712</td>
<td>131,087</td>
</tr>
<tr>
<td>3.00</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8712</td>
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</tr>
<tr>
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<td>9</td>
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<td>0</td>
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<td>0</td>
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<td>6.71</td>
<td>2.29</td>
<td>0</td>
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<td>2.29</td>
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<td>0.50</td>
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<td>2.29</td>
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<td>117,937</td>
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### 5. Conclusions and recommendations for future research

A model was developed that can determine the discount price to set a phased-out item so that the total initial inventory is depleted and revenue is maximized. The model was tested using sales records from a retail chain. Cluster analysis was applied to the data, and a nonlinear power regression model that reflects the price elasticity of demand was chosen to represent the relationship between sales volume and price. Furthermore, the different SKUs were compared to one another in order to study the relative price elasticity of demand. The case of selling the remaining inventory at the end of the phase-out horizon to a third party wholesaler was directly addressed and the threshold salvage price beyond which it is profitable to sell has been determined. In addition, the problem was looked at from a different perspective, where the discount prices were pre-determined, and the question to be answered was which discount prices should be used and for how many periods during the phase-out time horizon.

Deterministic models were presented in this paper for phased-out items. Further research needs to be performed by relaxing some of the simplifying assumptions. These research extensions include: (1) demand dependency of SKUs on each other using bundled constraints, (2) customers’ reactions to fluctuation in prices,